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SHORT COMMUNICATION

Zernike coefficients for concentric, circular scaled pupils: an equivalent expression

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We present an alternative formal calculation of the scaled Zernike coefficient expansion by means of the inner product of the Zernike polynomials and the wavefront error corresponding to the scaled pupil. The relationship exhibited by the radial polynomials and Bessel functions leads to a general expression in terms of the Gauss hypergeometric function. Direct properties and index selection rules are established, and easy derivation of the non-normalized coefficients is also straightforward.

Keywords: optical aberrations; mathematical physics

1. Introduction

The optical quality of an imaging system can be evaluated by studying the shape of the wavefront error at the exit pupil [1,2]. Since that wavefront error contains all the information concerning the aberrations of the system, the importance of its determination is a primary objective. Today, by means of interferometric testing, different wavefront sensing techniques, and wavefront error fitting algorithms, the objective is achieved [3]. The wavefront error is usually determined by using an orthonormal polynomial expansion. Individually or in groups, these can be seen as descriptors of the type of system aberration. The set of orthonormal polynomials widely adopted is that of the Zernike circle polynomials for systems having circular pupils. However, a method to obtain a set of orthonormal polynomials for other pupil characteristics and/or geometries inscribed in a unit circle has been published recently [4,5].

A great deal of work has been done in using the Zernike polynomials to characterize the imaging quality of optical systems. Nevertheless, in order to use this set of polynomials to quantify the wavefront error, a circular domain of unit radius in the pupil must be set. Goldberg and Geary [6] studied the problem of extrapolating the wavefront error beyond the domain established to fit it. That is, they estimated the full pupil wavefront error from its fit based on subdomains, which is usual in interferometric optical testing. Thus, they developed a matrix relation between the coefficients of the two expansions: those for the whole unit pupil from those of the smaller one established. Although, mathematically this latter expansion is not valid and it is unstable beyond the domain in which it is defined, as occurs in scaling or decentration subdomains [7], Golberg and Geary [6] established the conditions to find the minimum size of the subdomain in order for the wavefront error extrapolation to the full pupil to be valid within a given error.

The inverse problem is easier and mathematically less problematic: to find the Zernike coefficient expansion for a smaller pupil than that corresponding to the whole unit circle pupil. This is particularly important when the values of the aberrations need to be compared when they are obtained with different pupil radii, as occurs in ocular aberrometry, or when it is necessary to determine the aberration values for different smaller pupil diameters quickly. Many works have dealt with this problem, and some authors propose analytical methods [7–9], while others propose numerical ones [7,10–13], considering concentric, decentred, and even non-circular, scaled pupils. Both methodologies are important since they complement each other.

However, some comments are necessary. Decentered scaled pupils should be inside the unit circle pupil, since this guarantees that the new pupil will not contain part of the wavefront that is not described from the whole unit circle. If a portion of the scaled pupil extends beyond the unit circle, the new Zernike polynomial expansion might not be convergent and the errors should be quantified [6,8]. When the scaled pupil has a non-circular shape, the new set of polynomials derived from the Zernike set for the circular pupil needs to be demonstrated in order to

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formulate an orthonormal set of functions [5]. If not, errors in the coefficients arise and propagate in the variance, root-mean-square wavefront error, or Strehl ratio calculations.

Several of the aforementioned works have proposed analytical expressions for the new Zernike coefficients corresponding to the scaled pupil, with these expressions having a recursive nature [8] or being formulated in an intuitive way avoiding recursion relationships [9]. Furthermore, a mathematical demonstration of this latter approach has been published [14].

In this work, a direct and formal approach, developed for the theory of function expansion in terms of orthonormal polynomials, is used. Thus, we formulate an alternative and equivalent analytical expression for the Zernike coefficients of a concentric scaled circle pupil from the expressions corresponding to that of the whole unit circle. Since the wavefront error can be determined for the scaled pupil having a smaller radius, it can be expanded again in terms of the Zernike polynomials set considering the scaled pupil to have a unit radius. Then, the new coefficients are determined by the inner product of the corresponding Zernike polynomial with the wavefront error evaluated in the scaled pupil. The relationship exhibited by the radial polynomials and Bessel functions leads to a general expression in terms of the Gauss hypergeometric function. Direct properties and index selection rules are established avoiding cumbersome algebra, recursive relationships, and initial conditions.

This method has recently been applied to establish a quite simple expression in terms of the radial polynomials themselves [15], but no demonstration of the equivalence with previous results has been provided. Here, we will also present the equivalence of our results. Moreover, the expression for the non-normalized coefficients is straightforward.

2. Zernike circle polynomials and wavefront expansion

The standard Zernike circular polynomials are a product of radial polynomials, $R_n^m(r)$, and angular functions, $\Theta^m(\theta)$, as [16]

$$Z_n^m(\rho,\theta) = \sqrt{\frac{2(n+1)}{1+\delta_{m,0}}} R_n^m(\rho)\Theta^m(\theta) = N_n^m R_n^m(\rho)\Theta^m(\theta) \tag{1}$$

where N_n^m is the normalization constant, and $\delta_{m,0}$ the Kronecker delta symbol. Regarding the normalization constant, it is usual in interferometric optical testing not to include it.

The $\{Z_n^m\}$ forms a complete set of functions in the unit circle. The double-indexing scheme is used so that each Zernike polynomial will unambiguously have the highest power, n, of the radial polynomial, and the azimuthal frequency m of the angular function.

The radial polynomials are define as

$$R_n^m(\rho) = \sum_{s=0}^{(n-|m|)/2} \frac{(-1)^s (n-s)!}{s!\left(\frac{n+m}{2}-s\right)!\left(\frac{n-m}{2}-s\right)!}\rho^{n-2s} \tag{2}$$

and the angular function as

$$\Theta^m(\theta) = \begin{cases} \cos(m\theta), & m \geq 0, \\ \sin(m\theta), & m < 0. \end{cases} \tag{3}$$

The indices n and m must satisfy $m \leq n$ and $n-m \geq 0$ is an even number. We will use this double-indexing scheme in the present work, although the ordering of the Zernike polynomials into a single one is possible [5,17].

If $W(\rho,\theta)$ is the wavefront error of an optical system, i.e. the aberrations, in the exit pupil, it can be expanded in terms of the complete set of Zernike polynomials, $\{Z_n^m\}$, with the pupil radius normalized to unity, as

$$W(\rho,\theta) = \sum_{n=0}^{k} \sum_{m=-n}^{n} a_n^m Z_n^m(\rho,\theta) \tag{4}$$

where a_n^m is the coefficient of the corresponding basis element of the set, and k is the highest order of the radial polynomials adopted for the expansion.

3. Expansion coefficients

We will derive in this section the new coefficients for the wavefront error expansion in the scaled pupils from those corresponding to the whole unit pupil. Thus, if the radius of the unit pupil is multiplied by a normalized scale parameter $\epsilon = r/R$, where r is the new physical radius of the pupil and R is the original one, thus satisfying $0 \leq \epsilon < 1$, the wavefront error $W_\epsilon(\epsilon\rho,\theta)$ within this scaled pupil will be

$$W_\epsilon(\epsilon\rho,\theta) = \sum_{n=0}^{k} \sum_{m=-n}^{n} a_n^m Z_n^m(\epsilon\rho,\theta) \tag{5}$$

where Equation (4) has been applied.

Now, if we wish to expand this wavefront error in the Zernike polynomials set, assuming that the scaled pupil has a unit radius, it can be re-expressed as

$$W(\rho,\theta) = \sum_{n'=0}^{k} \sum_{m'=-n'}^{n'} b_{n'}^{m'} Z_{n'}^{m'}(\rho,\theta) \tag{6}$$

where $W_\epsilon(\epsilon\rho,\theta) = W(\rho,\theta)$, and the coefficients $b_{n'}^{m'}$ can be calculated by definition through the inner product of the corresponding Zernike polynomial, $Z_{n'}^{m'}(\rho,\theta)$ and the wavefront $W_{\epsilon}(\epsilon\rho,\theta)$, as follows:

$$b_{n'}^{m'} = \langle W|Z_{n'}^{m'}\rangle = \tfrac{1}{A}\textstyle\int_{\Sigma} W Z_{n'}^{m'}\,\mathrm{d}\mathbf{S} \tag{7}$$

where we have ignored the variables for brevity, A is the area of the pupil, and Σ the domain of integration of the pupil. Then

$$b_{n'}^{m'} = \tfrac{1}{\pi}\textstyle\int_0^1\int_0^{2\pi} W_{\epsilon}(\epsilon\rho,\theta) Z_{n'}^{m'}(\rho,\theta)\rho\,\mathrm{d}\rho\mathrm{d}\theta \tag{8}$$

given that $W_{\epsilon}(\epsilon\rho,\theta) = W(\rho,\theta)$.

If we take into account Equation (5) and the definition of the Zernike polynomials (1), we get

$$\begin{aligned} b_{n'}^{m'} &= \frac{1}{\pi}\int_0^1\int_0^{2\pi} W_{\epsilon}(\epsilon\rho,\theta) R_{n'}^{m'}(\rho)\Theta^{m'}(\theta)\rho\,\mathrm{d}\rho\mathrm{d}\theta \\ &= \frac{1}{\pi}\sum_{n=0}^{k}\sum_{m=-n}^{n}\sqrt{\frac{2(n+1)}{1+\delta_{m,0}}}\sqrt{\frac{2(n'+1)}{1+\delta_{m',0}}}a_n^m \\ &\quad\times\int_0^1 R_n^m(\epsilon\rho)R_{n'}^{m'}(\rho)\rho\,\mathrm{d}\rho\int_0^{2\pi}\Theta^m(\theta)\Theta^{m'}(\theta)\mathrm{d}\theta. \end{aligned} \tag{9}$$

The integral for the angular functions is solved as they are orthogonals,

$$\int_0^{2\pi}\Theta^m(\theta)\Theta^{m'}(\theta)\mathrm{d}\theta = \pi(1+\delta_{m,0})\delta_{m,m'}, \tag{10}$$

and thus we get

$$\begin{aligned} b_{n'}^{m} &= (1+\delta_{m,0})\sqrt{\frac{2(n'+1)}{1+\delta_{m,0}}}\sum_{n=0}^{k}\sqrt{\frac{2(n+1)}{1+\delta_{m,0}}}a_n^m \\ &\quad\times\int_0^1 R_n^m(\epsilon\rho)R_{n'}^m(\rho)\rho\,\mathrm{d}\rho \\ &= \sqrt{2(n'+1)}\sum_{n=0}^{k}\sqrt{2(n+1)}a_n^m\int_0^1 R_n^m(\epsilon\rho)R_{n'}^m(\rho)\rho\,\mathrm{d}\rho. \end{aligned} \tag{11}$$

We see from the above expression that the azimuthal frequency dependency indicates that the new coefficients, $b_{n'}^m$, are calculated by means of the previous coefficients, a_n^m, for the same value of m.

In order to solve the integral corresponding to the product of the radial functions, we first recall the definition of the radial polynomials in terms of the Bessel functions [18] given by the following expression:

$$R_p^q(\rho) = (-1)^{(p-q)/2}\int_0^{\infty} J_{p+1}(r)J_q(r\rho)\mathrm{d}r, \quad 0 \le \rho < 1. \tag{12}$$

Then, substituting for the radial polynomial $R_n^m(\epsilon\rho)$ in the integral above,

$$\begin{aligned} b_{n'}^{m} &= \sqrt{2(n'+1)}\sum_{n=0}^{k}\sqrt{2(n+1)}a_n^m(-1)^{(n-m)/2} \\ &\quad\times\int_0^1 R_{n'}^m(\rho)\rho\,\mathrm{d}\rho\int_0^{\infty} J_{n+1}(r)J_m(\epsilon r\rho)\mathrm{d}r \\ &= \sqrt{2(n'+1)}\sum_{n=0}^{k}\sqrt{2(n+1)}a_n^m(-1)^{(n-m)/2} \\ &\quad\times\int_0^{\infty} J_{n+1}(r)\mathrm{d}r\int_0^1 R_{n'}^m(\rho)J_m(\epsilon r\rho)\rho\,\mathrm{d}\rho \end{aligned} \tag{13}$$

where we have inverted the order of the integral in the variables r and ρ.

Now, we have the well-known result from Zernike–Nijboer diffraction theory of images [16,18],

$$\int_0^1 R_n^m(\rho)J_m(v\rho)\rho\,\mathrm{d}\rho = (-1)^{(n-m)/2}\frac{J_{n+1}(v)}{v}, \tag{14}$$

which, when applied to the later integral in r, leads to the following expression:

$$\begin{aligned} b_{n'}^{m} &= \sqrt{2(n'+1)}\sum_{n=0}^{k}\sqrt{2(n+1)}a_n^m(-1)^{\frac{n+n'-2m}{2}} \\ &\quad\times\int_0^{\infty} J_{n+1}(r)\mathrm{d}r\frac{J_{n'+1}(\epsilon r)}{\epsilon r} \\ &= \sqrt{2(n'+1)}\sum_{n=0}^{k}\sqrt{2(n+1)}a_n^m(-1)^{\frac{n+n'-2m}{2}} \\ &\quad\times\int_0^{\infty}\frac{J_{n+1}(r)J_{n'+1}(\epsilon r)}{\epsilon r}\mathrm{d}r. \end{aligned} \tag{15}$$

Finally, the integral in r can be solved in terms of the Gauss hypergeometric function, ${}_2F_1(a,b;\,c;\,\epsilon^2)$, taking into the tabulated results from [19]:

$$\begin{aligned} b_{n'}^{m} &= \sqrt{2(n'+1)}\sum_{n=0}^{k}(-1)^{\frac{n+n'-2m}{2}}\sqrt{2(n+1)}a_n^m \\ &\quad\times\frac{1}{\epsilon}\frac{\epsilon^{n'+1}\Gamma\left(\frac{n+n'}{2}+1\right)}{2\Gamma(n'+2)\Gamma\left(\frac{n-n'}{2}+1\right)}\,{}_2F_1 \\ &\quad\times\left(\frac{n'-n}{2},\frac{n+n'}{2}+1;n'+2;\epsilon^2\right) \\ &= \epsilon^{n'}\sqrt{(n'+1)}\sum_{n=0}^{k}(-1)^{\frac{n+n'-2m}{2}}\sqrt{(n+1)}a_n^m \\ &\quad\times\frac{\Gamma\left(\frac{n+n'}{2}+1\right)}{\Gamma(n'+2)\Gamma\left(\frac{n-n'}{2}+1\right)}\,{}_2F_1\left(\frac{n'-n}{2},\frac{n+n'}{2}+1;n'+2;\epsilon^2\right). \end{aligned} \tag{16}$$

In this result, we must remember that the Gauss hypergeometric function is convergent in the unit circle [20,21], since $n'+2>0$ for all n', and $c-a-b=1$. Furthermore, we should bear in mind that $n-m\geq 0$ and $n-m$ is even, and $n'-m'\geq 0$ and $n'-m'$ is an even number. Therefore, it follows that $n-n'$ is even. Moreover, an additional condition arises: the value for $n-n'$ must be different from any negative even integer in order for $\Gamma((n-n')/2+1)$ to be defined. Thus, we can rewrite the equation in the following way:

$$b_{n'}^{m}=\epsilon^{n'}\sum_{n=n'}^{k}(-1)^{\frac{n+n'-2m}{2}}\sqrt{(n'+1)}\sqrt{(n+1)}a_{n}^{m}\times\frac{\Gamma\left(\frac{n+n'}{2}+1\right)}{\Gamma(n'+2)\Gamma\left(\frac{n-n'}{2}+1\right)}\,{}_2F_1\left(\frac{n'-n}{2},\frac{n+n'}{2}+1;n'+2;\epsilon^2\right), \tag{17}$$

in which $n=n',\ n'+2,\ldots,k$.

Table 1 shows the expressions between the coefficients of the scaled pupils and the whole unit pupil, for the first six orders ($k=6$). The agreement with those tabulated by Schwiegerling [8] is fulfilled.

The main straightforward result is the expression for the non-normalized coefficients. For this, we need to take into account only the fact that the normalization constant is not present in the Zernike polynomials, and therefore in the inner product of the wavefront, $W_\epsilon(\epsilon\rho,\theta)$, and the corresponding orthogonal Zernike polynomial, $Z_{n'}^{m'}(\rho,\theta)$:

$$b_{n'}^{m'}=\frac{2(n'+1)}{\pi(1+\delta_{m',0})}\int_0^1\int_0^{2\pi}W_\epsilon(\epsilon\rho,\theta)R_{n'}^{m'}(\rho)\Theta^{m'}(\theta)\rho\,\mathrm{d}\rho\mathrm{d}\theta. \tag{18}$$

Thus, we get

$$b_{n'}^{m}=(n'+1)\,\epsilon^{n'}\sum_{n=n'}^{k}(-1)^{\frac{n+n'-2m}{2}}a_{n}^{m}\times\frac{\Gamma\left(\frac{n+n'}{2}+1\right)}{\Gamma(n'+2)\Gamma\left(\frac{n-n'}{2}+1\right)}\,{}_2F_1\left(\frac{n'-n}{2},\frac{n+n'}{2}+1;n'+2;\epsilon^2\right). \tag{19}$$

It is also important to check the results from Equation (17) when $\epsilon=1$ in order to recover the original coefficients in the wavefront expansion. In this case, we have to use the value of ${}_2F_1(\frac{n'-n}{2},\frac{n+n'}{2}+1;n'+2;1)$, and the properties of the Gamma function [20,21], Γ, to get finally

$$b_{n'}^{m}=\sqrt{(n'+1)}\sum_{n=n'}^{k}(-1)^{\frac{n+n'-2m}{2}}\sqrt{(n+1)}a_{n}^{m}\times\frac{2\sin\left(\frac{\pi}{2}(n-n')\right)}{\pi(n-n')(2+n+n')}, \tag{20}$$

which gives $b_n^m=a_n^m$ when $n=n'$, since for all other cases the terms in the summation are zero, regardless of the value for m.

4. Equivalence with previous published expressions

4.1. Dai's expression

To demonstrate that our result given by Equation (17) is equivalent to those reported previously [8,9], we need to take several things into account. First, we should bear in mind that $(n'-n)/2$ is zero or a negative integer, and thus the Gauss hypergeometric function is a polynomial [20,21]. Therefore,

Table 1. Coefficients $b_{n'}^{m}$ for the scaled pupil in terms of those corresponding to the whole unit pupil, up to sixth order ($k=6$).

n'	m	$b_{n'}^{m}$
0	0	$a_0^0+a_2^0\sqrt{3}(\epsilon^2-1)-a_4^0\sqrt{5}(2\epsilon^4-3\epsilon^2+1)+a_6^0\sqrt{7}(5\epsilon^6-10\epsilon^4+6\epsilon^2-1)$
1	1,−1	$\epsilon[a_1^m+a_3^m\sqrt{8}(\epsilon^2-1)+a_5^m\sqrt{3}(5\epsilon^4-8\epsilon^2+3)]$
2	−2, 0, 2	$\epsilon^2[a_2^m+a_4^m\sqrt{15}(\epsilon^2-1)+a_6^m\sqrt{21}(3\epsilon^4-5\epsilon^2+2)]$
3	−3, −1, 1, 3	$\epsilon^3[a_3^m+a_5^m 2\sqrt{6}(\epsilon^2-1)]$
4	−4, −2, 0, 2, 4	$\epsilon^4[a_4^m+a_6^m\sqrt{35}(\epsilon^2-1)]$
5	−5, −3, −1, 1, 3, 5	$\epsilon^5 a_5^m$
6	−6, −4, −2, 0, 2, 4, 6	$\epsilon^6 a_6^m$

$$b_{n'}^m = \epsilon^{n'}\sqrt{(n'+1)}\sum_{n=n'}^{k}(-1)^{\frac{n+n'-2m}{2}}\sqrt{(n+1)}a_n^m \times \frac{\Gamma\left(\frac{n+n'}{2}+1\right)}{\Gamma(n'+2)\Gamma\left(\frac{n-n'}{2}+1\right)}\frac{\Gamma(n'+2)}{\Gamma\left(\frac{n'+n}{2}+1\right)\Gamma\left(\frac{n'-n}{2}\right)} \times \sum_{j=0}^{(n-n')/2}\frac{\Gamma\left(\frac{n'-n}{2}+j\right)\Gamma\left(\frac{n'+n}{2}+j+1\right)}{\Gamma(n'+2+j)j!}\epsilon^{2j}$$

$$= \epsilon^{n'}\sqrt{(n'+1)}\sum_{n=n'}^{k}(-1)^{\frac{n+n'-2m}{2}}\sqrt{(n+1)}a_n^m \times \frac{1}{\Gamma\left(\frac{n-n'}{2}+1\right)\Gamma\left(\frac{n'-n}{2}\right)} \times \sum_{j=0}^{(n-n')/2}\frac{\Gamma\left(\frac{n'-n}{2}+j\right)\Gamma\left(\frac{n'+n}{2}+j+1\right)}{\Gamma(n'+2+j)j!}\epsilon^{2j}. \qquad (21)$$

Secondly, we note that if p is a positive integer, the following is verified [20,21]:

$$\frac{\Gamma(p-z)}{\Gamma(-z)} = (-1)^p\frac{\Gamma(z+1)}{\Gamma(z-p+1)}. \qquad (22)$$

Moreover, this can be applied to the last expression with $z \equiv (n-n')/2$ to obtain

$$b_{n'}^m = \epsilon^{n'}\sqrt{(n'+1)}\sum_{n=n'}^{k}(-1)^{\frac{n+n'-2m}{2}}\sqrt{(n+1)}a_n^m \times \sum_{j=0}^{(n-n')/2}(-1)^j\frac{\Gamma\left(\frac{n-n'}{2}+n'+1+j\right)}{\Gamma\left(\frac{n-n'}{2}-j+1\right)\Gamma(n'+2+j)j!}\epsilon^{2j}. \qquad (23)$$

We can arrange the expression a slightly more by applying the definition of the Gamma function [20]:

$$b_{n'}^m = \epsilon^{n'}\sum_{n=n'}^{k}(-1)^{\frac{n-n'}{2}+n'-m}\sqrt{(n'+1)(n+1)}a_n^m \times \sum_{j=0}^{\frac{n-n')}{2}}(-1)^j\frac{\left(\frac{n-n'}{2}+n'+j\right)!}{\left(\frac{n-n'}{2}-j\right)!(n'+1+j)!j!}\epsilon^{2j}. \qquad (24)$$

Finally, given that n has values from n', $n'+2,\ldots$ to k, introducing the index i, and rearranging the expression, it follows that

$$b_{n'}^m = \epsilon^{n'}\sum_{i=0}^{(k-n')/2}\sqrt{(n'+1)(n'+2i+1)}a_{n'+2i}^m \times \sum_{j=0}^{i}(-1)^{j+i}\frac{(n'+i+j)!}{(i-j)!(n'+1+j)!j!}\epsilon^{2j}. \qquad (25)$$

This is the equation reported by Dai [9] in his work. It is not difficult to follow this procedure to show the equivalence for the non-normalized coefficients (Equation (19)) to those reported by Dai in the same work.

4.2. Janssen and Dirksen expression

Janssen and Dirksen [15] reported an expression for the non-normalized scaled coefficients in terms of the Zernike radial polynomials which is quite concise:

$$b_{n'}^m = \sum_{n=n'}^{k}a_n^m\left[R_{n'}^n(\epsilon) - R_{n'}^{n+2}(\epsilon)\right] \qquad (26)$$

in which k is the maximum order. From the definition of radial Zernike polynomials, it is not difficult to derive the following relationship for the above equation:

$$b_{n'}^m = \sum_{n=n'}^{k}(n'+1)a_n^m\sum_{s=0}^{(n-n')/2}(-1)^s \times \frac{(n-s)!}{\left(\frac{n-n'}{2}-s\right)!\left(\frac{n+n'}{2}+1-s\right)!\,s!}\epsilon^{n-2s}. \qquad (27)$$

Now, we can reorder the indices in the summations,

$$b_{n'}^m = (n'+1)\sum_{i=0}^{(k-n')/2}a_{n'+2i}^m \times \sum_{s=0}^{i}(-1)^s\frac{(n'+2i-s)!}{(i-s)!(n'+i+1-j)!\,s!}\epsilon^{n'+2i-2j}, \qquad (28)$$

and finally with $s=i-q$,

$$= \epsilon^{n'}(n'+1)\sum_{i=0}^{(k-n')/2}a_{n'+2i}^m \times \sum_{q=0}^{i}(-1)^{i-q}\frac{(n'+i+q)!}{(i-q)!(n'+q+1)!\,q!}\epsilon^{2q}. \qquad (29)$$

This is in full agreement with the expression obtained by Dai [9], and easily obtained following the procedure of the previous subsection by starting from the non-normalized coefficients derived in this work (Equation (19)).

5. Summary

We have presented an alternative, and equivalent, expression for the Zernike coefficient expansion corresponding to a concentric, scaled circular pupil, in terms of that for the whole unit pupil. The direct method by means of the inner product of the wavefront and the Zernike polynomials avoids cumbersome algebra, recursive relationships, and initial conditions. The use of hypergeometric functions allows the coefficients to be evaluated numerically, although some modern computer algebra software (e.g. Mathematica, from Wolfram Research Inc.) can manage them.

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References

[1] Shannon, R.R. *The Art and Science of Optical Design*; Cambridge Press: New York, 1997.
[2] Mahajan, V. *Optical Imaging and Aberrations: Ray Geometrical Optics*; SPIE Press: Bellingham, WA, 1998.
[3] Malacara, D. Ed. *Optical Shop Testing*, 3rd ed.; John Wiley & Sons: New York, 2007.
[4] Mahajan, V. *Optical Imaging and Aberrations: Wave Diffraction Optics*; SPIE Press: Bellingham, WA, 2001.
[5] Mahajan, V.; Dai, G. *J. Opt. Soc. Am. A* **2007,** *24,* 2994–3016.
[6] Goldberg, K.A.; Geary, K. *J. Opt. Soc. Am. A* **2001,** *18,* 2146–2152.
[7] Dai, G. *Wavefront Optics for Vision Correction*; SPIE: Bellingham, WA, 2007.
[8] Schwiegerling, J. *J. Opt. Soc. Am. A* **2002,** *19,* 1937–1945.
[9] Dai, G. *J. Opt. Soc. Am. A* **2006,** *23,* 539–543.
[10] Campbell, C.E. *J. Opt. Soc. Am. A* **2003,** *20,* 209–217.
[11] Bará, S.; Arines, J.; Ares, J.; Prado, P. *J. Opt. Soc. Am. A* **2006,** *23,* 2061–2066.
[12] Lundström, L.; Unsbo, P. *J. Opt. Soc. Am. A* **2007,** *24,* 569–577.
[13] Comastri, S.A.; Perez, L.I.; Pérez, G.D.; Martin, G.; Bastida, K. *J. Optic. Pure Appl. Optic*. **2007,** *9,* 209–221.
[14] Shu, H.; Luo, L.; Han, G.; Coatrieux, J.L. *J. Opt. Soc. Am. A* **2006,** *23,* 1960–1966.
[15] Janssen, A.J.E.M.; Dirksen, P. *J. Microlith. Microfab. Microsyst*. **2007,** *5,* 030501-1–3.
[16] Born, M.; Wolf, E. *Principles of Optics*, 8th ed.; Cambridge Press: New York, 2002.
[17] Thibos, L.N.; Applegate, R.A.; Schwiegerling, J.T.; Webb, R.; Members, V.S.T. *OSA Trends in Optics and Photonic News, Vision Science and its Applications*. **2000,** *35,* 232–244.
[18] Nijboer, B.R.A. *Physica*. **1947,** *13,* 605–620.
[19] Gradshteyn, I.; Ryzhik, I. *Table of Integrals, Series, and Products*, 7th ed.; Academic Press: New York, 2007.
[20] Abramowitz, M.; Stegun, I.A. *Handbook of Mathematical Functions*; Dover Inc: New York, 1965.
[21] Erdelyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F. *Higher Trascendental Functions*; McGraw–Hill: New York, 1955; Vol. 1.
[22] Wolfram, S. *The Mathematica Book*, 5th ed.; Wolfram Media Inc: Champaign, IL, 2003.

Appendix 1. Example of using Equation (17)

This appendix illustrates how the main result of this paper (Equation (17)) can be used to obtain the Zernike coefficient value for the scaled pupil from that corresponding to the full pupil. Thus, let us consider a full pupil wavefront error given by the sum of primary spherical aberration, coma, and astigmatism, each with a standard deviation of unity. Then $W(\rho,\theta)$ can be expressed in terms of the orthonormal Zernike polynomials as

$$W(\rho,\theta) = \sqrt{6}\rho^2\cos(2\theta) + \sqrt{8}(3\rho^3 - 2\rho)\cos(\theta) + \sqrt{5}(6\rho^4 - 6\rho^2 + 1)$$

in which $a_2^2 = a_3^1 = a_0^0 = 1$, and W has wavelength units. All the remaining coefficients a_n^m are zero independently of the values of the indices n and m as well as the maximum order k of the radial polynomials used for the expansion.

Let us suppose that we want to determine the coefficients for the scaled pupil with a reduction factor $\epsilon = 0.8$. By using Equation (17), the value of the non-zero scaled coefficients in wavelength units are

$$\begin{aligned} b_0^0 &= a_0^0\frac{\Gamma(1)}{\Gamma(2)\Gamma(1)}\,{}_2F_1(0,1;2;0.8^2) \\ &\quad - \sqrt{3}a_2^0\frac{\Gamma(2)}{\Gamma(2)\Gamma(2)}\,{}_2F_1(-1,2;2;0.8^2) \\ &\quad + \sqrt{5}a_4^0\frac{\Gamma(3)}{\Gamma(2)\Gamma(3)}\,{}_2F_1(-2,3;2;0.8^2) \\ &\quad + \cdots = 0 + 0 - 0.225396, \\ b_1^1 &= 0.8\left[2a_1^1\frac{\Gamma(2)}{\Gamma(3)\Gamma(1)}\,{}_2F_1(0,2;3;0.8^2)\right. \\ &\quad \left. + 2\sqrt{2}a_3^1\frac{\Gamma(3)}{\Gamma(3)\Gamma(2)}\,{}_2F_1(-1,3;3;0.8^2) + \cdots\right] = 0 - 0.814587, \\ b_2^0 &= 0.8^2\left[3a_2^0\frac{\Gamma(3)}{\Gamma(4)\Gamma(1)}\,{}_2F_1(0,3;4;0.8^2)\right. \\ &\quad \left. + \sqrt{15}a_4^0\frac{\Gamma(4)}{\Gamma(4)\Gamma(2)}\,{}_2F_1(-1,4;4;0.8^2) + \cdots\right] = 0 - 0.892336, \end{aligned}$$

$$\begin{aligned} b_2^2 &= 0.8^2\left[3a_2^2\frac{\Gamma(3)}{\Gamma(4)\Gamma(1)}\,{}_2F_1(0,3;4;0.8^2)\right. \\ &\quad \left. + \sqrt{15}a_4^2\frac{\Gamma(4)}{\Gamma(4)\Gamma(2)}\,{}_2F_1(-1,4;4;0.8^2) + \cdots\right] = 0.64 + 0, \\ b_3^1 &= 0.8^3(a_3^1 + \cdots) = 0.512, \\ b_4^0 &= 0.8^4(a_4^0 + \cdots) = 0.4096. \end{aligned}$$

The calculations show that it is advantageous to take into account the relationship ${}_2F_1(-a, b; b; z) = (1-z)^{-a}$, as well as the Gauss relations for contiguous functions [20,21]. For example, the non-zero term in the calculation of b_0^0 can be determined easily as follows:

$$\begin{aligned}
&{}_2F_1(-2, 3; 2; 0.8^2) \\
&= \frac{1}{2(1-0.8^2)}\left[2 \cdot {}_2F_1(-2, 2; 2; 0.8^2)\right. \\
&\quad \left. -4 \cdot 0.8^2 \cdot {}_2F_1(-2, 3; 3; 0.8^2)\right] = -0.1008,
\end{aligned}$$

which multiplied by $\sqrt{5}$ gives $b_0^0 = -0.225396$.

Appendix 2. Mathematica Code for listing the scaled coefficients

In this appendix, we provide as an example the Mathematica [22] code which lists the values for the scaled coefficients in terms of those corresponding to the full pupil up to any order:

(*Here Equation (17) is defined*)

$b[\mathrm{order\_}, np\_, m\_, \epsilon\_] := \mathrm{Module}[\{n, \mathrm{coeff}\},$

$\mathrm{coeff} = \mathrm{Sum}[a_{n,m} * \sqrt{(n+1)} * \sqrt{(np+1)} *$

$(-1)^{\frac{(n-2m+np)}{2}} * \epsilon^{np}\,\mathrm{Gamma}\left[\frac{1}{2}(2+n+np)\right]$

Hypergeometric2F1Regularized

$\left[\frac{1}{2}(-n+np), \frac{1}{2}(2+n+np), 2+np, \epsilon^2\right]$

$/\left(\mathrm{Gamma}\left[\frac{1}{2}(2+n-np)\right]\right),$

$\{n, np, \mathrm{order}, 2\}]];$

(*Here we set the maximum order of Zernike polynomials, up to the 20th polynomial as an example*)

order=20;

(*Then we list the coefficients*)

$\mathrm{Table}[\mathrm{FullSimplify}[\mathrm{Collect}[\mathrm{Expand}[b[\mathrm{order}, np, m, \epsilon]],$

$\sqrt{\_\_\_}a\_\_\_], (np-m)/2 \in \mathrm{Integers},$

$\mathrm{ComplexityFunction} \rightarrow \mathrm{LeafCount}],$

$\{\mathrm{np}, 0, \mathrm{order}\}]//\mathrm{TableForm}$